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Bifurcation analysis of multiple-attractor flight dynamics

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Studies of global aircraft dynamics, using nonlinear methods such as bifurcation analysis, reveal the influence of the various steady-state attractors on system behaviour. For obvious reasons, the vast majority of stability and control studies concentrate on achieving adequate performance and flying qualities on the 'trim branch' of the aircraft—the attractor on which conventional level flight, climbing and turning manoeuvres are centred. Investigations into other branches of attractors are usually limited to spin characteristics and spin recovery.

It is conceivable that future aircraft designs, combining agility with low observability, will exhibit less classical dynamic features than are currently typical. Furthermore, the provision of high levels of control power over a wide range of flight conditions (via new control motivator technology) may allow the existence of multiple attractors to be exploited by the pilot and/or control system.

This paper reports on a study of how bifurcation analysis can be deployed in this manner. In particular, the concept of 'tailoring' of bifurcations by the design team, in order to utilize the existence of multiple attractors, is described. Centre manifold/eigenstructure concepts form the basis of the proposed methodology. These are illustrated by application to an aircraft model in which scheduling of a single-axis thrust vectoring control effector is used to create a codimension-2 bifurcation, dramatically modifying dynamics at high angles of attack.

Future developments of the technique are discussed briefly, as is the important related issue of computation of basins of attraction for stable attractors.

Keywords: nonlinear flight dynamics; bifurcation analysis; codimension-2 bifurcations; bifurcation tailoring; nonlinear control law

1. Introduction

This paper reports on the development of a new approach to the stability and control analysis of fighter-type aircraft. It concerns the application of bifurcation analysis to aircraft flight-dynamics models, paying particular attention to the multiple-attractor nature of the flight envelope. The methodology utilizes centre-manifold concepts to manipulate the system bifurcationary behaviour, and generates direct links to the design variables that engineers may use for this purpose. The benefits to industry lie in the potential for this nonlinear-dynamics approach to contribute to improved aircraft and control systems design.

The basis of the approach is to provide the engineer with tools that enable the fundamental nonlinear nature of global aircraft models to be related, in a meaningful way, to the design parameters. The premise is that post-stall nonlinear features of the aircraft system can no longer be regarded as undesirable operating areas to be avoided, but rather that they are very much part of the medium with which the engineer must mould a workable design.

In § 2, the paper deals briefly with the justification for developing such an approach, by considering trends in fighter-aircraft design. Thereafter, in § 3, the scope of the bifurcation-analysis methodology is outlined, including the aspects of nonlinear-dynamics analysis typically applied in flight-mechanics models. In § 4, the sample aircraft system is described, and its behaviour discussed, by using representative one-parameter bifurcation diagrams. This is followed in § 5 by an exposition of the proposed design methodology, with sample results; a case is made for the generalized use of the method. A short validation of the results is given in § 6. Conclusions are drawn in § 7 and attention is drawn to potential for further research.

2. Justification for multiple-attractor analysis

The nonlinear nature of flight dynamics has been recognized for most of the history of heavier-than-air flight. It was also realized, however, that aircraft could be optimized on the basis of the required design specifications such that they would exhibit essentially linear behaviour in the intended operational envelope. Thus the techniques developed for analysis of aircraft stability and control evolved from simplified linear models originating in the early part of this century.

This linear operating region was bounded by nonlinear phenomena (such as stall, usually followed by spin) that were preferably to be avoided, justifiably so, since the pilot usually has minimal control over the vehicle under such conditions. Hence, many of the developments accompanying the rapid progression of aeronautical sciences to their present maturity were able to rely extensively on linear small-perturbation methods. From a dynamics point of view, all operational conditions represented stable hyperbolic solutions to the equations of motion, and if the open-loop system approached any critical points (in the sense of the linear eigenvalues crossing to the right half-plane) then either a stability augmentation system would be introduced or some sort of advisory or enforced envelope-limiting mechanism would be implemented.

After World War II, with the advent of jet propulsion, the development of combat aircraft configured for supersonic performance revealed new problems in the field of flight mechanics, especially trying to extend capabilities associated with rapid manoeuvres and tight turns (crucial to success in air combat). It became necessary to invest heavily in finding solutions to these difficulties; in the process, the technologies associated with stability and control of combat aircraft have diverged quite considerably from those appropriate to the study of transport-type vehicles.

These new problems in flight dynamics arose mainly from the widening of the flight envelope such that nonlinear conditions were encountered more frequently, or with more severe consequences, than in the case of earlier aircraft. High-rate body-axis roll manoeuvres produced *kinematic coupling* with unacceptable lateral-directional effects. In the alternative velocity-vector rolls, and in other rapid motions, the tailoring of the vehicle geometric layout to accommodate high-speed flight led to *inertial* coupling nonlinearities resulting in very sudden loss of control. And, even during low rates of manoeuvre, the new swept-wing long-forebody layouts introduced nonlinearity and coupling into the *aerodynamic* reactions.

Whatever the fundamental cause, it became increasingly evident that nonlinear phenomena were starting to adversely impact on aircraft safety and operational effectiveness. In response to this, much has been achieved in fending off these nonlinear intrusions. The principle contributions have been in terms of

(i) stabilizing aerodynamic flows;

(ii) widening the operating region in which the aerodynamic control surfaces (the conventional means of piloted input to the system) remain effective;

(iii) improved high-integrity automatic flight-control systems (which allowed stability margins to be reduced so as to enhance manoeuvrability); and

(iv) the advent of unconventional control motivators, such as thrust vectoring, forebody vortex control, split-aileron and deflecting wing-tip yaw devices (usually coupled with high thrust-to-weight ratios).

It was, for example, essentially via (i) and (ii) above, that the Russian Sukhoi 27 aircraft performed the now famous 'Cobra manoeuvre' in 1989, the first time that an operational fighter aircraft was demonstrated in symmetric controlled flight at angles of attack peaking at *ca.* 100°. Such capabilities were unprecedented: angle of attack, α (also referred to in aeronautical jargon as angle of incidence), is the dominant physical flow property governing aerodynamic response; no aircraft had previously been flown in a controlled manner at values of α beyond the stall, or break, after which the vehicle usually departs into some undesirable mode (such as spin). This remains the case with most fighter aircraft currently in service, although many achieve agile and 'care-free handling' (at $\alpha < \alpha_{\text{break}}$) using advanced flight-control systems (item (iii) above); by extensive modification of their flight-control laws, some of these aircraft could probably execute some post-stall manoeuvres (Orlick-Rückemann 1992).

The tactical manoeuvring advantages of post-stall agility had been recognized much earlier than the first public display of the Cobra (Herbst 1983) and several projects were undertaken (notably in the US and in Germany) to exploit item (iv) in the above list. These new control methods represent the technology being incorporated in future fighter aircraft, the Lockheed Martin F-22 being the first production aircraft expected to use thrust-vectoring. Another feature of future military aircraft that is likely to further emphasize the need for nonlinear design techniques is that of stealth: the requirement for low observability. The Rockwell/DASA X-31 thrust-vectored experimental aircraft (with a flight envelope extending to $\alpha \approx 70°$) has been used to explore the possibility of deleting the tail fin from future designs (Alcorn *et al*. 1996) and the first such vehicle, the Boeing/NASA X-36, is currently under development.

Overall, the trends summarized above present a significant challenge to stability and control engineers (particularly when combined with suggestions that future combat aircraft may be pilotless, thus facilitating substantially greater manoeuvre demands than those able to be tolerated by humans). Rather than limiting the flight envelope to the moderate-α quasi-linear regime, studies will need to take account of the fact that there are, typically, multiple system attractors governing the flight dynamics within these far wider operating boundaries.

The idea of item (iv) listed above is to provide sufficient control power, using multiple motivators, so that an aircraft with suitably augmented stability will in

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fact continue to operate in an intuitive near-linear fashion right up to the $\alpha \approx 70^\circ$ region. However, the physics governing inertial, kinematic and flow-related nonlinearities, that are inherent within the system, will be unchanged: in order to design the necessary control systems, a global understanding of the open-loop dynamics of the vehicle will be required, including the influence of all the attractors that appear within the envelope limits and the local bifurcations separating the attractor branches. Moreover, there is a clear need for flight dynamicists to go beyond the conventional low-α approach and to develop an appreciation of the relationship between nonlinear dynamics and the physics that drives it.

These advances do, of course, encompass a wide range of engineering sciences. One of these which is also pertinent to flight dynamicists, is the means by which the aerodynamic reactions are incorporated in the equations of motion: unsteady effects may be prevalent during rapid manoeuvres and existing modelling methods are inadequate in capturing time dependencies. In a practical implementation of the technique described in § 5, the correct representation of transient response is essential in validating the results.

3. Scope of the study

Aircraft are flexible systems with response modes spanning a range of frequencies. It is usual to consider only the rigid-body degrees of freedom (DOF) when investigating stability and control, handling qualities and performance issues. This is justified, particularly in the case of highly manoeuvrable aircraft, due to the relatively stiff structure. Although the frequency difference between the flexible (aeroelastic) and rigid-body modes may in fact be less in newer configurations than in the past, the standard approach is adopted here.

In deriving the six-DOF equations of motion for the unconstrained vehicle, a coordinate system attached to a point fixed on the surface of the Earth is chosen as the inertial reference system, with both the curvature and rotation of the Earth ignored. The equations are written with reference to a fixed point on the aircraft (a reference centre of gravity or some other appropriate point). As shown in most flight-mechanics texts, this results in a set of 12 equations, the 12 states being three translational velocities, three rotational velocities, three orientation angles and three positions (with respect to the Earth reference). The aircraft is treated as a single rigid body: DOF arising from coupling with control-surface dynamics and from rotating masses in the propulsion system are neglected.

For stability and control studies, the position and heading angle relative to the Earth can be ignored; also, since altitude variations involve only relatively small changes in forces and moments during the short time-intervals of concern, height is regarded as constant. This leaves eight first-order ordinary differential equations that relate forces and moments to aircraft motion and orientation. Using a Cartesian-axes system with origin at the centre of mass:

$$I_{xx}\dot{p} = qr(I_{yy} - I_{zz}) + I_{xz}(\dot{r} + pq) + L, \tag{3.1 a}$$

$$I_{yy}\dot{q} = rp(I_{zz} - I_{xx}) + I_{xz}(r^2 - p^2) + M, \tag{3.1 b}$$

$$I_{zz}\dot{r} = pq(I_{xx} - I_{yy}) + I_{xz}(\dot{p} - qr) + N, \tag{3.1 c}$$

$$\dot{\alpha} = q - \tan\beta(p\cos\alpha + r\sin\alpha) + \frac{Z_w}{mV_{\mathrm{T}}\cos\beta}, \tag{3.1 d}$$

$$\dot{\beta} = p\sin\alpha - r\cos\alpha + \frac{Y_w}{mV_{\mathrm{T}}}, \tag{3.1 e}$$

$$\dot{V_{\mathrm{T}}} = \frac{X_w}{m}, \tag{3.1 f}$$

$$\dot{\phi} = p + q\sin\phi\tan\theta + r\cos\phi\tan\theta, \tag{3.1 g}$$

$$\dot{\theta} = q\cos\phi - r\sin\phi, \tag{3.1 h}$$

where I_{xx}, I_{yy} and I_{zz} are moments of inertia about the x-, y- and z-body axes; I_{xz} is a cross-product of inertia; L, M and N are the rolling, pitching and yawing moments (about the x-, y- and z-axes, respectively); m is aircraft mass; p, q and r are the roll, pitch and yaw rates (about the x-, y- and z-axes); V_{T} is total flight path velocity; X_w, Y_w and Z_w are the axial, side and normal forces relative to flight-path (wind) axes; α and β are the angles of attack and sideslip; and ϕ and θ are the bank and pitch orientation angles, respectively. L, M, N, X_w, Y_w, Z_w include aircraft weight and aerodynamic, control and propulsive loads.

Note that these equations assume that x–z is a plane of symmetry from the geometric and inertial point of view; aerodynamic, or other asymmetries in the applied loads, would be included within L, M, N, X_w, Y_w and Z_w. It is normal in stability and control work to regard aircraft mass and mass distribution as constant.

Equations (3.1) can be written in the generic form:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\delta}, t), \qquad \boldsymbol{x}, \boldsymbol{f} \in \mathbb{R}^n, \quad \boldsymbol{\delta} \in \mathbb{R}^m, \tag{3.2}$$

where $\boldsymbol{x}$ is a vector of n state variables (in this case $\begin{bmatrix} p & q & r & \alpha & \beta & V_{\mathrm{T}} & \phi & \theta \end{bmatrix}^{\mathrm{T}}$); $\boldsymbol{\delta}$ is a vector of m parameters (or controls); $\dot{\boldsymbol{x}}$ is the time derivative of $\boldsymbol{x}$; and $\boldsymbol{f}$ is a set of n nonlinear functions.

The most difficult challenge in representing a specific aircraft in such equations is to adequately model the aerodynamic reactions within L, M, N, X_w, Y_w, Z_w, and this is particularly true of fighter aircraft operating at high α and capable of rapid manoeuvres. The standard approach is that of 'stability coefficients': a first-order Taylor series formulation for each of the three forces and three moments in terms of state and control variables; nonlinearity is accounted for by tabulating the coefficients against one or more state or control variable(s). Unsteady influences are usually represented only by including derivatives with respect to $\dot{\alpha}$ (i.e. a limited instantaneous dependence which cannot account for the various time-scales associated with separated and vortical flows), although some coefficients are occasionally also tabulated against motion frequency.

Such ad hoc aerodynamic models cannot be rigorously justified due to the simplifications involved with both nonlinear coupling and time dependencies (Macmillen 1996*a*; Peskett & Lowenberg 1996; Greenwell 1997). Much research effort is being invested in alternative modelling techniques, but use of the stability coefficient formulation in the industry continues, due to the familiarity that engineers have when trying to relate the forces and moments to physically meaningful parameters. This form of aerodynamic model ensures that equation (3.2) is autonomous ($\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\delta})$).

In the context of the current work, bifurcation analysis involves solving for sets of attractors in parts of the phase-control space. For each computation, one of the parameters is selected as the 'free parameter', λ (where $\lambda = \delta_i$ for some $1 \leqslant i \leqslant m$), which is varied within some convenient range, while the other $(m-1)$ parameters are held constant. Branches of fixed-point attractors are found by solving

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \lambda) = \boldsymbol{0}, \qquad \boldsymbol{x} \in \mathbb{R}^n, \quad \lambda \in \mathbb{R}. \tag{3.3}$$

Limit cycle, or periodic-orbit, attractors are solutions of

$$\dot{\boldsymbol{x}}(t) = \dot{\boldsymbol{x}}(t+T) = \boldsymbol{f}(\boldsymbol{x}(t), \lambda), \qquad \boldsymbol{x}, \boldsymbol{f} \in \mathbb{R}^n, \quad T, \lambda \in \mathbb{R}, \tag{3.4}$$

where T is the period of the orbit.

Parametric continuation methods are used to solve equations (3.3) and (3.4) numerically. The computed attractor branches are presented in the form of one-parameter bifurcation diagrams, showing two-dimensional slices of the equilibrium surface in the form of a state variable versus the free parameter.

Two-parameter bifurcation diagrams—a projection of the equilibrium surface onto the parameter space—may also be generated, usually by regarding a second parameter as a state variable ($x_{n+1} = \delta_j$, $\delta_j \neq \lambda$) and adding an extra equation, $g(\boldsymbol{x}, \boldsymbol{\delta})$, to ensure that solutions are also bifurcation points. For example, $g(\boldsymbol{x}, \boldsymbol{\delta}) = \det \boldsymbol{F} = \boldsymbol{0}$, where $\boldsymbol{F}$ is the Jacobian matrix of $\boldsymbol{f}$ with respect to $\boldsymbol{x}$, will produce trajectories of all bifurcations in which a real eigenvalue of $\boldsymbol{F}$ is zero.

Two-parameter bifurcation diagrams have been used in flight-dynamics studies to delineate control-deflection combinations that avoid bifurcations (typically aircraft aileron–rudder interconnect systems). Powerful use has been made of these analysis techniques (inferring global aircraft behaviour characteristics based on attractors in state-control space) in high-α flight mechanics (Carroll & Mehra 1982; Lowenberg 1991; Guicheteau 1993*a*; Jahnke & Culick 1994; Macmillen 1996*b*; Patel & Smith 1996). The technique has also been used to explore aspects of control law design (Goman & Khramtsovsky 1995; Avanzini & de Matteis 1996; Littleboy & Smith 1997; Liebst & DeWitt 1997; Wang *et al*. 1997). An interesting application of bifurcation analysis in the post-stall regime evaluates the role of thrust vectoring in spin entry and recovery (Planeaux & McDonnell 1991), while another investigates regions of chaotic response precipitated by oscillatory control inputs (Gránásy *et al*. 1998).

The premise for using branches of attractors to infer aircraft behaviour is that λ is varied in a quasi-static manner (Thompson & Bishop 1994). Pilots, on the other hand, do not consciously fly with this in mind! This raises questions associated with the extent to which transient motions will or will not follow the expected attractor branches. Time-step integrations are usually used to investigate this (as well as to validate assumptions in the model, etc.). In recent years, efforts have been made to supplement the attractor branches with some estimate of their basins of attraction (Guicheteau 1993*b*; Jahnke & Chen 1995; Goman & Khramtsovsky 1997). Such stability-region information substantially enhances the confidence with which predictions of real system behaviour can be made, particularly in regions where attractors are expected to compete. Applications to robust control problems, for example, will benefit from knowledge of basin boundaries.

There is, however, ample evidence, such as from piloted simulations (Patel 1996), that one-parameter bifurcation diagrams are indeed effective 'maps' for aircraft behaviour all the way from low α through to spins. Such findings have proved important in justifying the use of bifurcation analysis in practical design applications.

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One of the advantages of adopting the bifurcation analysis technique is that it facilitates a very general approach and frees the designer from some of the traditional paradigms of linearized flight mechanics. The choice of which variables are designated 'state variables', and which is selected to be the 'free parameter', is up to the user: parameters such as control system gains can be designated as either free parameter or state variable, and motion variables such as p, q, r can be regarded as parameters if desired. This flexibility is central to the approach described in § 5.

The focus in the explanation of the proposed methodology is the scheduling of a control (parameter) at high α to modify the structural stability of the system. This is not full control-law design: control-system dynamics is not accounted for in the example model. (Note that time-lag effects in control systems or in unsteady aerodynamic modelling will only be reflected in periodic, quasi-periodic and chaotic attractors—not in point equilibria.)

4. The sample aircraft model

The aircraft model upon which the development of the method is based, is that of the McDonnell Douglas F-4J Phantom. Designed in the 1960s, this is not the type of post-stall capable fighter described in § 2. It is, however, convenient for this study for the following reasons.

1. The aerodynamic formulation is relatively simple.
2. The model is symmetric: there is zero rolling or yawing moment or side force when lateral/directional controls (aileron and rudder) are zero; while this is expected under normal flight conditions, it is well established that a major difficulty with newer long-forebody configurations is that micro-asymmetries set in at higher α, leading to large yawing and other reactions.
3. Although simple, the model has been validated as being representative of the nonlinear phenomena exhibited by the full-scale aircraft (Mitchell *et al*. 1980).
4. The behaviour of this aircraft has been the subject of extensive study, and the physical phenomena underlying the nonlinearities are well understood.

The issue of aerodynamic symmetry requires further explanation. Since some fighter configurations manoeuvring at post-stall incidences will inevitably experience asymmetric effects, it is essential that the methodology works under such conditions. For development purposes, however, the decoupling of lateral/directional reactions from longitudinal responses simplifies the eigenstructure interpretation.

The model itself is taken from Mitchell *et al*. (1980). The information from which the model was extracted was originally assembled from three separate data sources for the purposes of F-4J combat manoeuvring simulation. The model used is valid for a single altitude of 15 000 feet (4570 m), flaps, slats and undercarriage retracted and partly full internal fuel tanks. The aerodynamics is represented by a stability coefficient formulation, covering an incidence range of 0–110° and a sideslip range of −30 to +30°. It uses 24 coefficients, all tabulated against α, with one being a function of both α and β. The aerodynamic controls are aileron, rudder, spoiler and stabilator; but aileron and spoiler are mechanically linked (aileron deflects down on the up-going wing and spoiler deploys by a proportional amount on the upper surface of the down-going wing), so there are in fact three independent aerodynamic controls.

The F-4 suffers typical high-α phenomena for a 1970s configuration. On increasing incidence to *ca.* 19°, the aircraft begins an oscillation that manifests itself predominantly in the roll DOF. This phenomenon is known as *wing rock* and it signals that the 'Dutch-roll mode'—an oscillatory mode exhibited by all conventional aircraft—has become unstable. From an aircraft stability point of view, the trigger is a reduction in 'effective dihedral' (lateral stiffness), caused by adverse sidewash on the fuselage afterbody and horizontal tail induced by the wing-fuselage combination (Lowenberg 1991). From a nonlinear dynamics viewpoint, it is a Hopf bifurcation to a limit cycle (once the mode has become unstable and the oscillation commences, there typically exists a nonlinear damping characteristic such that the roll is undamped at low bank angles, ϕ, but becomes damped once ϕ builds up). The wing-rock amplitudes can reach $\pm 30°$ in ϕ and $\pm 10°$ in β.

As α increases beyond 20°, the vertical tail also enters the adverse sidewash field and the dynamic pressure in the tail region is reduced (due to shielding by the aft fuselage and the wake of the stalling wing). The resulting combination of reduced effective dihedral and negative weathercock stability induces *lateral-directional departure*. This is, in essence, an uncommanded 'nose slice', coupled with roll, that may occur without warning and takes the aircraft into post-stall gyrations leading to spin. It is usually regarded as a complete loss of lateral-directional static stability. A study of the wing-rock limit cycle seems to indicate that it coincides with a loss in stability of the periodic orbit at a torus bifurcation, when the incidence oscillations reach *ca.* 21–22°.

It should be noted that the departure occurs *prior* to the onset of aerodynamic stall, defined as the point of maximum lift. This is typically the case, leading to suggestions that onsets of nonlinearity be referred to as 'break-points' rather than 'stall' (Hancock 1995).

After departure, the aircraft is faced with multiple attractor branches. The ensuing response has the characteristics of a chaotic attractor (Lowenberg 1991) and appears to result from a strong interaction between two 'symmetrically' placed asymptotically unstable branches (symmetrical in the sense described above, namely having equal magnitude and opposite sign in respect of lateral-directional variables p, r, β and ϕ but identical values of longitudinal variables q, α, θ and V_{T}).

This discussion of aircraft response in terms of both bifurcation analysis terminology and also physical configuration/aerodynamic influences is essential to the justification for successful adoption of such analysis tools in the design environment. The F-4 behaviour and model is ideal for expressing these links without becoming embroiled in some of the complications of more complex systems. It is not only the effects of asymmetry in aircraft models that make the interpretation of results more difficult; the multi-dimensional tables associated with newer fighter-aircraft models lead to potential difficulties relating to smoothing of data (Lowenberg 1995; Macmillen 1996*b*).

The bifurcation analysis performed in this study uses an eighth-order model, as described in § 3. It is often the case that lower-order models are adopted, and this is quite valid for certain flight regimes. A fifth-order model, for example, in which V_{T}, θ and ϕ are kept constant, is suitable for situations where fluctuations in flight velocity are fairly slow and there are no rapid orientation changes that would induce significant gravitational nonlinearities. A seventh-order model, with V_T constant, is appropriate for representing manoeuvres where thrust is matched to drag to keep a more or less constant flight speed (e.g. a steady turn).

The above model order refers to open-loop ('airframe-only') models. In the case of the F-4, this is acceptable since the stability augmentation system (SAS) on the aircraft is of restricted authority. A limited study of a tenth-order version of the F-4J model incorporating the SAS shows that the high-α phenomena described above appear to be shifted only slightly in terms of α; bigger differences would be revealed in comparisons involving both lateral stick and pedal inputs. Some newer aircraft are designed with unstable bare-airframe characteristics so that only the closed-loop system (i.e. with full-authority flight-control system modelled) is stable; a bifurcation analysis has been applied to such an aircraft by Avanzini & de Matteis (1996).

When generating one-parameter bifurcation diagrams, a choice needs to be made of which parameter is to be varied (the rest remaining constant). When the aim is to depict standard global stability and control characteristics of the aircraft, it is usually one of the actual control variables—aileron, stabilator, rudder or thrust—that is chosen. In the examples shown in this paper, it is stabilator (δ_{stab}) that is used, while aileron and rudder are fixed at zero deflection. Hence, the quasi-steady manoeuvre reflected by the equilibrium branches is a symmetric pitch-up-type motion. In fact, high-α motion is usually utilized by pilots during *turning* manoeuvres: the successful outcome of close-in air-to-air combat usually favours the aircraft with the smallest turning radius; it is such turns that produce the demand for large values of lift which, in turn, requires high α. Nevertheless, the results obtained in the symmetric manoeuvre are similar in several respects to those obtained during turns.

The bifurcation analysis has been performed on the F-4J model by coding it into the author's FORTRAN continuation method program 'PCS' (which solves only for point attractors); output is plotted in MATLAB. The subroutine in which the model is specified, and the associated data file, are used (with minor modification) also for the code AUTO (Doedel *et al.* 1994) to compute periodic solutions (limit cycles). The model has also been set up within MATLAB's Simulink simulation package (MathWorks Inc. 1993) to permit bifurcation results to be compared with time-histories, i.e. in the presence of transient motions.

Figure 1 shows four projections of the one-parameter bifurcation diagram (point equilibria only) with δ_{stab} as the free parameter, $\delta_a = \delta_r = 0$. In all results shown, the thrust is fixed at $T = 60$ kN and natural spline interpolation is used for all tabular data (tensioned splines are recommended (Macmillen 1996*b*) but natural splines work well here due to extra data points having been added to the tables at an earlier stage, when linear interpolation was being used).

In all bifurcation diagrams, solid lines represent asymptotically stable point attractors (all eigenvalues of the system Jacobian matrix in the left half-plane) and dashed lines asymptotically unstable solutions (one or more eigenvalues in the right half-plane). In PCS-generated bifurcation diagrams, the circle denotes a Hopf bifurcation (real part of a complex conjugate eigenvalue pair crossing through zero). In AUTO plots, circles represent the maximum amplitude of limit cycles, filled circles being stable (eigenvalues of Poincaré mapping within the unit circle), open circles unstable (at least one such linear eigenvalue has modulus greater than unity).

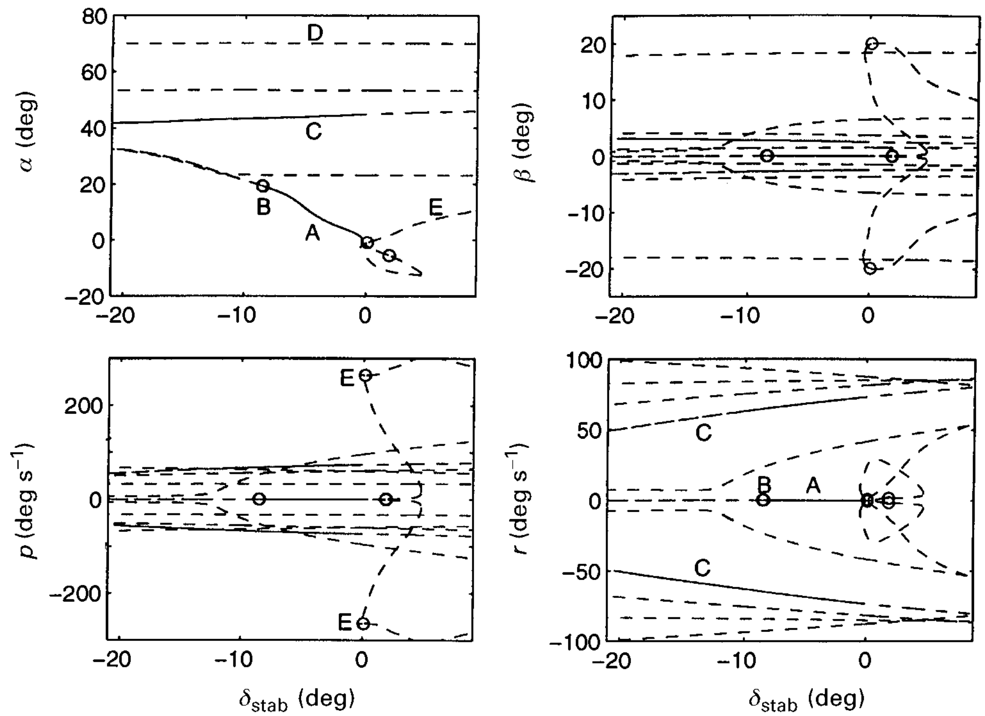

Figure 1. Baseline F-4J model equilibria (the continuation parameter is stabilator deflection δ_{stab}; solid lines represent stable equilibria, dashed lines asymptotically unstable solutions; circles are Hopf bifurcations).

Since the development of the new design methodology in § 5 requires a solid understanding of the baseline F-4J dynamics, the following points concerning the labelled branches of equilibria in figure 1 should be noted:

A is the 'trim branch'—the attractor governing all conventional flight;

B is the Hopf bifurcation to the wing-rock limit cycle;

C if it were a stable point (or periodic) attractor would be a spin branch, designated 'steep spin' due to the relatively low values of α;

D like C, this would represent spin—'flat spin' in this case since α is nearer the 90° normally associated with flat spin;

E is a region of *autorotation*—where non-zero rolling occurs at low incidence, due to combined aerodynamic and inertial loads (this phenomenon is studied in detail in Lowenberg & Champneys (this issue)).

There is a further branch at $\alpha \approx -55°$ (inverted spin) which is omitted from figure 1 for the sake of clarity; it is also asymptotically unstable.

We note that none of the post-departure branches is stable. Thus, little can be said of the likely spin mode that the aircraft would enter under such conditions. However,

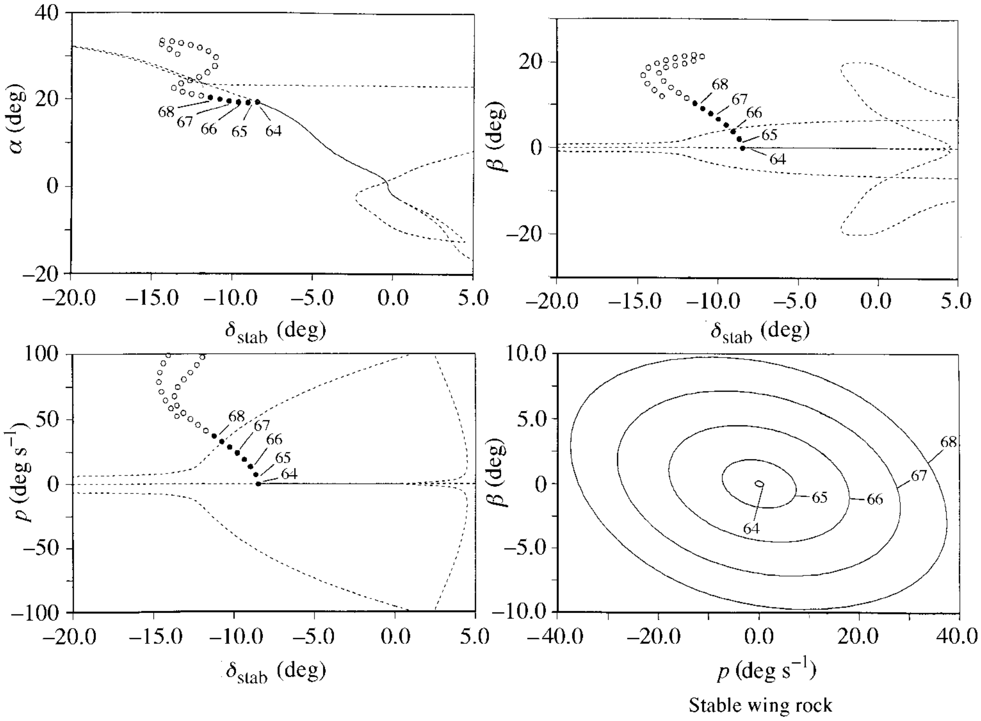

Figure 2. Baseline F-4J model: wing-rock periodic orbits (one-parameter bifurcation diagrams for α, β and p are shown, and a β–p phase diagram showing growth in limit-cycle amplitude corresponding to points 64–68 on the bifurcation diagrams).

as mentioned above, time-history runs reveal a recurrent motion that seems chaotic—and the bifurcation diagrams help to form an understanding of the mechanisms underlying this.

Figure 2 shows one-parameter bifurcation diagrams for α, β and p, reflecting some of the same branches as in figure 1 but including values of the wing-rock limit-cycle maximum amplitude (generated in AUTO). A phase portrait is also shown, depicting the growth of the stable part of the attractor as δ_{stab} changes. The nature of the limit cycle is evident from the various projections of the bifurcation diagram, the roll rate being predominant. The Hopf bifurcation from the stable equilibria occurs at $\delta_{\text{stab}} = -8.5°$, $\alpha = 19.4°$. At $\delta_{\text{stab}} = -11.8°$ the limit cycle loses stability at a torus bifurcation. Although further analysis of the ensuing dynamics would be of interest, numerical time-step integrations reveal that the system departs at this point into a spin mode.

We shall next describe a stability-manipulation design methodology and demonstrate its usefulness on the aspects of the F-4J dynamics illustrated in figures 1 and 2.

5. Bifurcation-manipulation methodology

As observed in § 4, the F-4J model loses its ability to maintain symmetric flight at $\delta_{\text{stab}} = -11.8°$ when α reaches values of 21–22°. We suppose here that the designers of the aircraft wish to extend its operating envelope to higher α, and that this is to be achieved by scheduling of control deflections. We assume also that there is the possibility of an additional pitch-axis control device to provide control power at high angles of incidence when the stabilator loses efficiency; supplementary control power in the lateral/directional sense is, however, considered too expensive to implement.

This simulated problem is a credible one if we ignore the fact that an aircraft designed for controlled post-stall manoeuvres would be configured slightly differently in an aerodynamic sense (to maintain symmetric flow to higher incidence). The additional pitch-axis control motivator is hypothetical: it is modelled as a thrust-vectoring system (with nozzle located 6 m aft of the centre of mass, pitch-deflection travel $\delta_\zeta \pm 22°$). Such a system is feasible for most fighter aircraft; but the F-4 nozzles are located on the fuselage such that upward swivelling would be impractical—we ignore this and regard the model as some generic aircraft with aft-mounted nozzles. The thrust vectoring is implemented within the F-4J model purely in an ideal manner (no thrust losses due to vectoring, thrust line angle equals nozzle angle).

The conventional way of extending the incidence to which the model can operate in a stable fashion is essentially what is done in the actual F-4J SAS: since lateral/directional stability is lost, it is these DOF that are augmented—by feeding back the sideslip angle β (or equivalent) to the rudder deflection. However, the mechanism that destroys directional stability also affects the rudder: both aileron and rudder have low control effectiveness in this incidence range. An earlier study (Lowenberg 1996) shows that the level of stability augmentation that is needed is unattainable.

The problem, therefore, is to find some alternative way of extending the operating region.

The new methodology is described here with respect to this specific example, with the corresponding generalized interpretation also being highlighted. The five steps are as follows.

1. *Identify possible high-α operating region.* It is possible that the lateral-directional instability that is at the core of the problem recovers stable attributes once the incidence has increased to the point where the tail emerges from the wing wake. Therefore, we investigate behaviour at *higher* α. Figure 3 shows a selection of one-parameter bifurcation diagram projections as in figure 1 but with the free parameter being $k\delta_{\text{stab}}$ ($k = 4$, selected arbitrarily so as to give a wide range of α). An alternative is simply to run the continuation with δ_{stab} running from, say, -80 to $30°$. Either approach implies, for the F-4 model, an artificial augmentation of longitudinal control power but no other influence on the aerodynamic model. It is evident that there is indeed a new stable symmetric attractor branch at $40° \leqslant \alpha \leqslant 53°$. It is also clear that the other high-α point attractor branches remain asymptotically unstable.

If no new stable region had been found in this simple manner, then the use of other parameters would be indicated. In this case, the use of δ_{a} or δ_{r} would introduce asymmetric motions, which could be acceptable in certain circumstances; but here, we are looking for a symmetric-flight solution. It is possible, therefore, that in the absence of a new stable branch no solution to the problem as specified can be found. It is also possible, however, for some unstable solutions at high α to be a result of longitudinal mode instability, and therefore likely to be amenable to stabilization via feedback.

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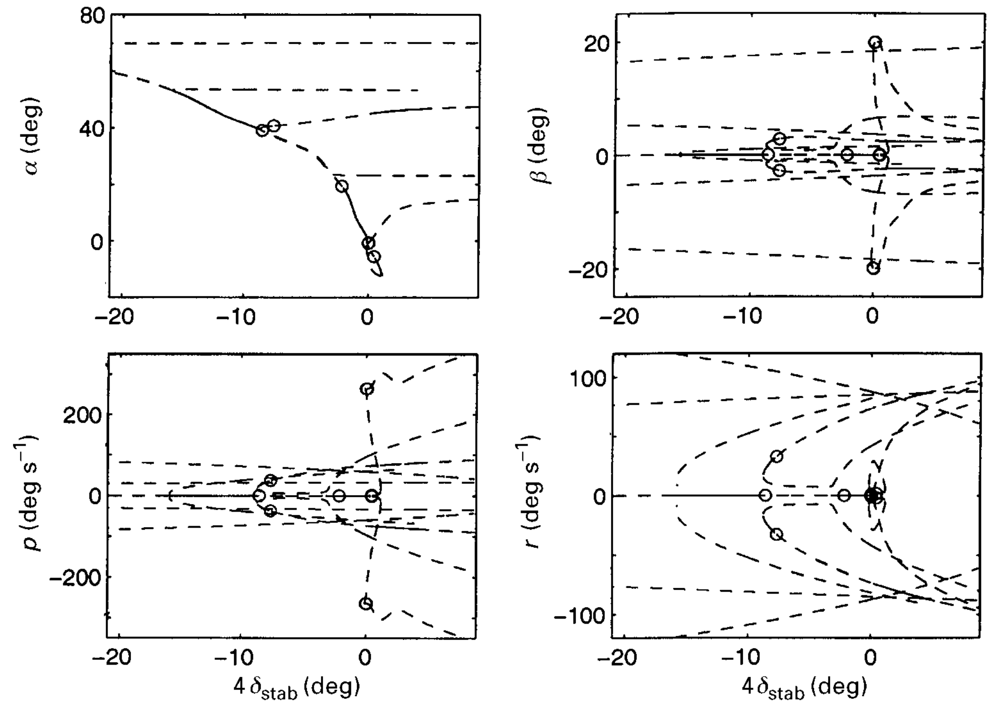

Figure 3. F-4J model equilibria: the free parameter is $4\delta_{\text{stab}}$.

2. *Identify physical means of accessing new operating region.* Having found a 'target operating regime', a route to this new branch must be devised. This entails modifying the bifurcationary behaviour of the system such that the new high-α branch shifts to within the physically attainable δ_{stab} range, and such that the route from the low-α symmetric branch to the high-α attractor is acceptable. Thus a means of manipulating the phase-control space needs to be found.

If, in the general case, the location of the target operating regime is a strong function of several state and control variables, then it may be difficult to devise an acceptable route to get there. In the F-4 example, a consideration of the physics of the problem shows that the new high-incidence stable branch is dependent principally on α: if one had sufficient pitch-control power to reach $\alpha > 40°$ *within* the stabilator range of $-21° \leqslant \delta_{\text{stab}} \leqslant 9°$, this new attractor could be incorporated within the flight envelope.

The second step, therefore, is to provide the supplementary control power: in this case by pitch-thrust vectoring, as described above. Once this is specified, its impact on system dynamics can, of course, be observed by solving the equations as in figure 1 but with the nozzle angle, δ_{ζ}, as a free parameter. Conversely, the results illustrated in figure 3 can be used to determine the capability that the vectoring system must have in order to be able to trim the aircraft at $\alpha > 40°$.

3. *Identify route to new region.* The next problem is to develop a route, on the bifurcation diagrams, for getting from the low-α trim branch to the high-α condition in an acceptable way. It is important to allow the system to operate to as high an incidence on the low-α branch as possible (for conventional manoeuvres), without it departing to spin.

It is necessary, therefore, to analyse the departure mechanism. In the present example, we know that it is a lateral-directional departure and we know that rudder and aileron are severely limited in power at these incidences. The general case requires some modal studies to be conducted in the region at which behaviour becomes unacceptable (the region of the first bifurcation). In our case, to avoid complications associated with the wing-rock periodic orbit, we choose the Hopf bifurcation (at $\delta_{\text{stab}} = -8.5°$, $\alpha = 19.4°$) to be the undesirable bifurcation that must be avoided (it is, in fact, possible to let the aircraft oscillate until just before the torus bifurcation which leads to spin entry).

By the 'centre manifold theorem', bifurcationary behaviour may be studied in terms of a reduced-order system of the same dimension as the number of eigenvalues of the linear problem that are in the vicinity of the imaginary axis. The PCS program can be configured to calculate a number of 'auxiliary variables', including eigenvalues and eigenvectors, during the continuation process. Figure 4 shows the evolution of the smallest real eigenvalue (of Jacobian matrix $\boldsymbol{F}$) and the real part of the complex eigenvalue closest to the imaginary axis. (Rather than tracking a specific eigenvalue, only that with smallest magnitude of real part is plotted, so the 'discontinuities' are merely a result of eigenvalues changing magnitude as the branch evolves; additional eigenvalues can be included when necessary.)

Figure 5 shows the eigenvector of the complex eigenvalue as it crosses the imaginary axis at the Hopf bifurcation. This serves merely to illustrate the relative magnitude of the mode of motion on the state variables: in this case the lateral-directional variables dominate (the only visible longitudinal state is pitch rate q). This is expected, due to the well-understood symmetric dynamics of the F-4J model. But, in a general case, there may be strong coupling between variables; then simple modal diagrams such as this inform the engineer which states play the least dominant role in the undesirable bifurcation.

In this case, it seems that if a bifurcation in the longitudinal (symmetric) plane could be created, by altering stability in respect of the longitudinal variables, immediately prior to the Hopf point, the lateral-directional ('undesirable') bifurcation may not arise. The new bifurcation must involve responses that are acceptable, in this case, symmetric. In a relatively simple example such as that outlined here, conventional design methods could be used to show that it is longitudinal static stability ($\mathrm{d}M/\mathrm{d}\alpha$) that must be lost in order to achieve a symmetric ('pitch-up') bifurcation.

In the general sense, once it has been established (i) *where* the acceptable bifurcation is to occur; (ii) the *plane* to which it should be restricted (in the present example, corresponding to the aircraft plane of symmetry); and (iii) the *parameter to be used* to induce the bifurcation and direct it appropriately (δ_ζ here), then some means of specifying this parameter's values at the new bifurcation is needed.

Where there are a number of parameters and it is less clear which one to use in point (iii) above, a study such as that depicted in figure 6 may be useful. During the continuation process, the derivative of eigenvalues with respect to parameters has been computed at each point, helping to identify which control is likely to provide the required phase-control space modification most conveniently. This is a novel aspect of flight-mechanics analysis, in that the sensitivity to control variables is related directly to overall system stability via the eigenvalues of the linearized model. (As a result of the simple numerical differencing algorithm used, the results shown in figure 6 contain significant numerical noise.)

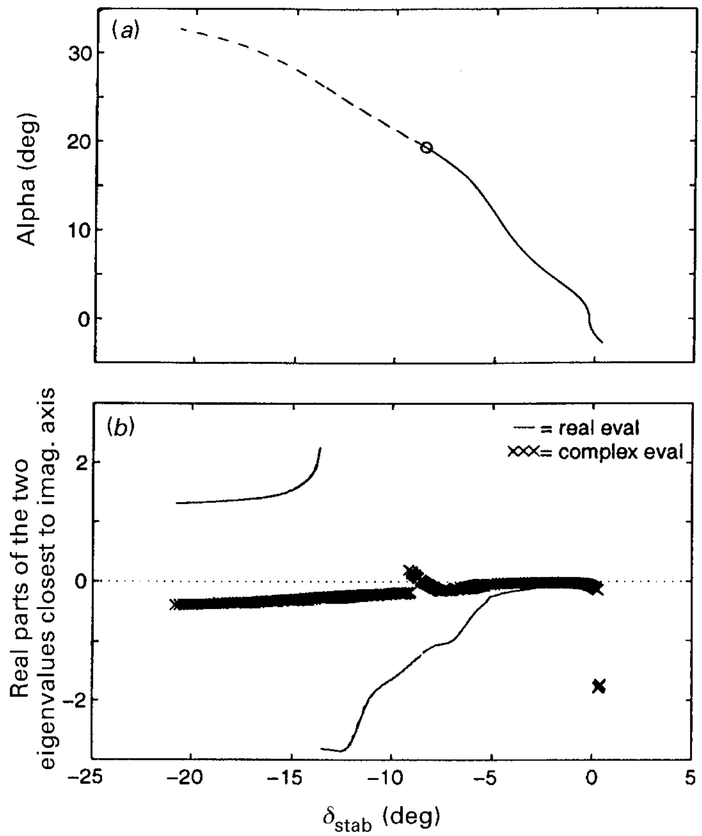

Figure 4. F-4J model equilibria: variations of smallest eigenvalues for symmetric branch ((*a*) α versus δ_{stab}, is part of the symmetric branch in figure 1; (*b*) evolution of smallest real eigenvalue and real part of complex eigenvalue with smallest real part, for this branch as δ_{stab} changes).

4. *Define control law.* Clearly, it is a fold-type bifurcation that is most likely to be useful in bridging the two stable attractor regions. What is now desired, in fact, is to determine the parameter variations necessary to join the 'end-point' of the existing attractor (immediately prior to the Hopf point in the example) to the 'starting point' of the other (high-α) attractor. This is the crux of the present methodology.

Referring to figure 4, one approach that could be envisaged is to extend the nth-order system by adding an equation that requires the magnitude of the smallest real eigenvalue to be moved to and kept at zero. The new parameter (δ_ζ) is defined as the $(n+1)$th state variable. This approach does produce solutions: the value of the eigenvalue at the starting point needs to be known (from the nth-order system results) and the user specifies some region over which it is to be 'dragged' to zero as the continuation method progresses. In this case, however, there is no nearby fold bifurcation and the system traverses rapidly to a totally different portion of the state-control space. If, once the eigenvalue has reached zero it is kept there, then a two-parameter bifurcation diagram is traced out. Although not useful in this instance, this approach shows how continuation methods can be exploited to provide parameter variations that achieve user-defined eigenvalue positioning. Difficulties

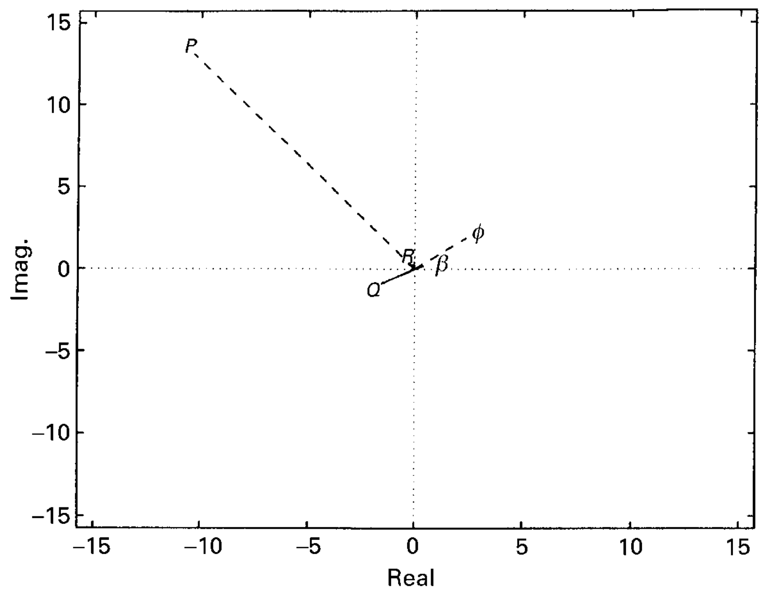

Figure 5. F-4J model equilibria: eigenvector corresponding to critical eigenvalue $-0.008 \pm i5.51$ (near Hopf bifurcation on symmetric branch). The α, θ and V_T components are too small to be visible.

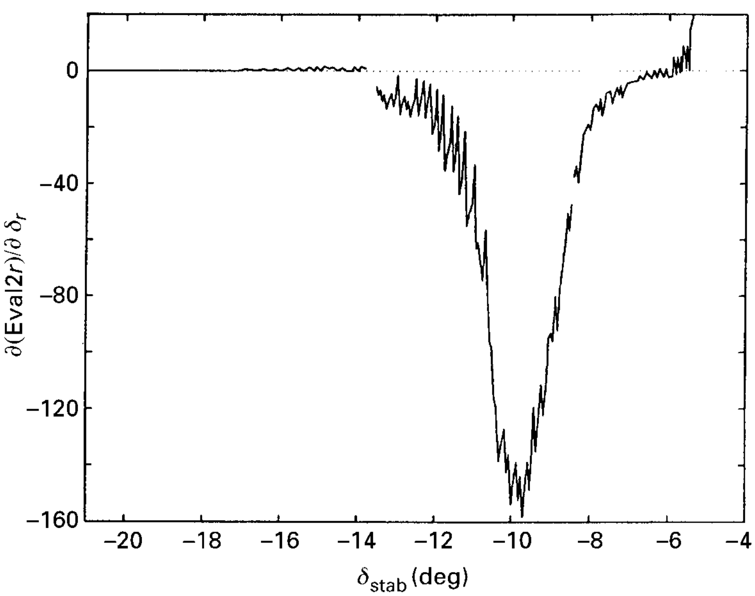

Figure 6. F-4J model equilibria: sensitivity of second-smallest real eigenvalue to δ_r for symmetric branch, over a range of δ_{stab} values.

can arise, however, in the tracking of a *specific* eigenvalue, as the eigenstructure may change appreciably along a solution branch.

The method adopted here, referred to as 'bifurcation tailoring', is as follows: if the supplementary pitch-control device (δ_ζ) has the required power (to trim at $\alpha > 40°$)

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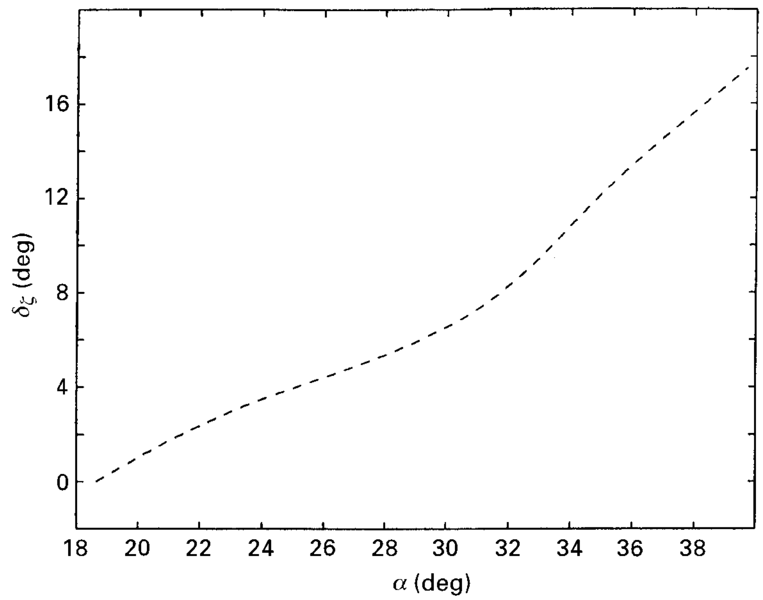

Figure 7. F-4J model equilibria: δ_ζ variations to join the two stable symmetric branches.

then it should be possible for δ_{stab} and δ_ζ to be blended in various ways to allow transition from low- to high-α trim. This transition must (i) avoid lateral-directional bifurcations; and (ii) ensure that stable symmetric trim exists for all achievable δ_{stab} and δ_ζ values. The desired bifurcation to the high-α trim must therefore be initiated prior to the limit cycle becoming unstable (we choose $\delta_{\text{stab}} = -8°$ here, at which $\alpha = 18.6°$), and the $\alpha \geqslant 40°$ branch must be shifted, relative to its location in figure 3, so that its stable region exists for $\delta_{\text{stab}} \geqslant -8°$; in order to avoid hysteresis, we shall require the high-α portion of the branch to become stable at exactly $\delta_{\text{stab}} = -8°$.

This is to be achieved by devising a schedule for δ_ζ in terms of α. The schedule is obtained by fixing the original free parameter, δ_{stab}, at $-8°$; α is defined as the new free parameter, and δ_ζ is treated as a state variable (so the system remains eighth-order). This modified system was run in the PCS continuation code over the range $18.6° \leqslant \alpha \leqslant 40°$. The resulting variation in pitch nozzle deflection is shown in figure 7.

To generalize the process in this step, we change the original system (equation (3.3)) to the new nth order system $\dot{\boldsymbol{y}} = \boldsymbol{h}(\boldsymbol{y}, \lambda')$, where $\boldsymbol{y}$ is the same as the original state vector $\boldsymbol{x}$ except that one element, x_i, is removed and replaced with the control variable to be scheduled (i.e. $y_i = \delta_k$). Equilibria are computed with x_i as the free parameter ($\lambda' = x_i$), while the original free parameter, λ, is now prescribed (either fixed, in which case $\boldsymbol{h} = \boldsymbol{f}$, or given some convenient variation with one or more state variables). The resulting solution provides the desired schedule of δ_k in terms of x_i. We have already discussed the choice of thrust-vector angle δ_ζ for δ_k in the present example; the selection of α as x_i is because it is a variable within the plane chosen for the modified behaviour: we wish to vary δ_ζ with α to create a new plane-of-symmetry bifurcation.

5. *Implement control law.* Step 4 provides a schedule for parameter δ_ζ that should create the exact bifurcation needed when implemented within the original system

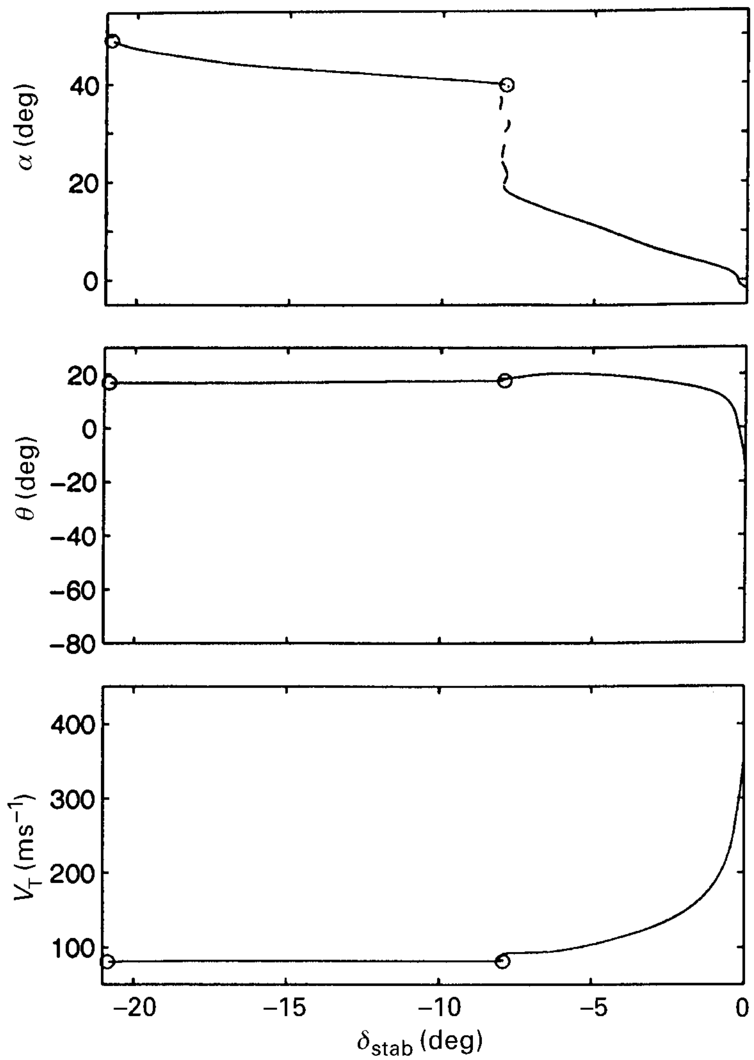

Figure 8. F-4J model equilibria: new symmetric branch with δ_ζ scheduled.

(i.e. the eighth-order system with δ_{stab} as a free parameter, α a state variable). Figure 8 confirms the existence of the desired symmetric bifurcation (in this case δ_ζ continues slightly above the 17.84° indicated at $\alpha = 40°$ in figure 7, thus giving stable behaviour up to almost $\alpha = 50°$).

The objective of the analysis has been achieved: the engineer can use this δ_ζ schedule directly, or adapt it in some suitable way.

The new bifurcation is of codimension 2 with respect to the original system: two control parameters are needed in order for it to exist. The near-vertical line from $\alpha = 18.6°$ to $\alpha = 40°$, on the α–δ_{stab} equilibrium diagram, is not a degenerate fold bifurcation: no real eigenvalues are zero for any part of the solution branch shown. There is a discrete change in the eigenvalues (of the original system) at the start and end-points of the transition region. The aircraft's transient response through this

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region can therefore be expected to be fast, even if δ_{stab} is moved only marginally beyond $-8°$.

In reality, we have a 'piecewise-continuous' situation in that the original system exists prior to the new bifurcation while a *modified* system (with the scheduling) is used for $\alpha \geqslant 18.6°$. With the scheduling considered as intrinsic to the governing equations, we require just *one* free parameter to generate figure 8.

The bifurcation with δ_{stab} fixed is a special case. It is also possible to define a variation of δ_{stab} with, for example, α; this allows a variety of control blending laws to be created and evaluated, and the process can be extended to codimension three and beyond (for example, when modifying a spin branch, sensible rudder deflections may be envisaged in addition to the longitudinal control variations).

6. Validation

From figure 8 it is evident that the new bifurcation entails a very large α traverse, although changes in the other motion variables are negligible. It implies, therefore, significant transient motion which could cause difficulties in capturing the high-α equilibrium.

Figure 9 shows a time-history in which δ_{stab} is varied from -2 to $-18°$ (i.e. nose up/stick pulled back); a small aileron disturbance is provided as a trigger for lateral-directional effects. The outcome is favourable in that the transition to symmetric flight at $\alpha \geqslant 40°$ occurs rapidly, with negligible overshoot. As may be expected, however, with thrust held constant (to match the bifurcation runs) the flight velocity, V_{T}, drops considerably with the increased drag at high incidence. (In the simulation shown in figure 9, a simple pitch-rate feedback to δ_{ζ} was deployed after the transition between low and high α; this reduced the longitudinal excursions as the aircraft maintains trimmed flight through the low-speed region.)

The magnitude of transient response was, if anything, exacerbated in this time-history by varying δ_{stab} and δ_{ζ} simultaneously. In order to obtain the bifurcation corresponding to figure 8, δ_{stab} should be fixed while $18.6° < \alpha < 40°$. Nevertheless, figure 9 confirms the ability to fly the aircraft at $\alpha \geqslant 40°$, at least under some circumstances.

Despite this satisfactory result, this does not constitute a full analysis. Figure 10 shows all the equilibria branches in the flight envelope, using the δ_{ζ} schedule. It is evident that the steep spin branch remains close (particularly in terms of α) to the new high-α region, and part of it is now stable; there is a danger that the aircraft could transition to steep spin instead of symmetric flight. This is clearly a situation in which knowledge of the basins of attraction for the various branches would be of considerable benefit. In the absence of such stability region computations, the robustness of the new symmetric branch to control deflections and transient motions needs to be investigated. This has been done to a limited extent in Lowenberg (1997), using time-history simulations and a two-parameter bifurcation diagram.

The outcome of that study is that, although the new branch can be used in the presence of moderate asymmetric disturbances, there is indeed cause for concern regarding its small basin of attraction relative to the nearby steep-spin branches. The entire bifurcation manipulation process can be repeated on the steep-spin branches, this time looking to move the folds evident in figure 10 further to the right-hand

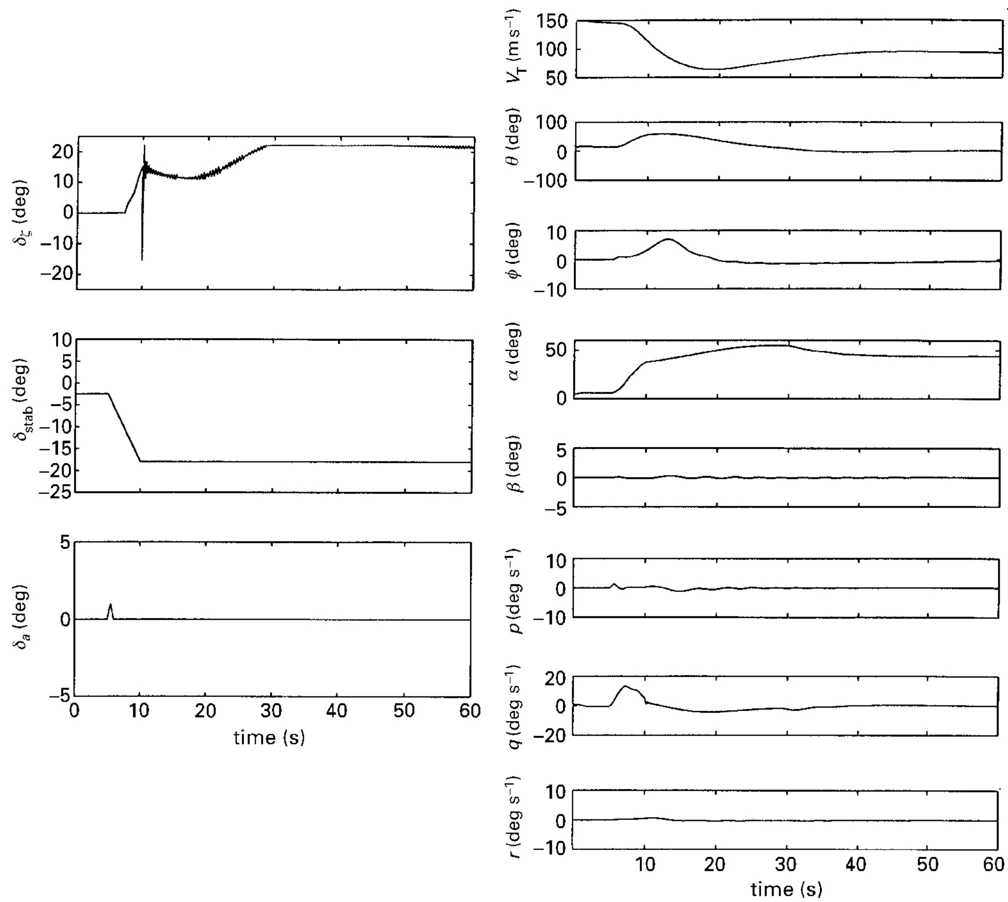

Figure 9. F-4J eighth-order model time-history: δ_ζ scheduling used with δ_{stab} input to pitch up to high-α symmetric trim branch.

side of the plot. This has in fact been achieved by using the bifurcation tailoring approach described in § 5 (Lowenberg 1998).

7. Conclusions

Future fighter-type aircraft are certain to make more use of the nonlinear operating region of the flight envelope than do current configurations. In order to design the airframe, propulsion system and control laws to provide stable flight up to high angles of attack it will be necessary to understand the underlying dynamics over a range of different system attractor regions. Furthermore, it may be advantageous for the control laws to be designed so as to in fact *alter* the system dynamics.

A novel control scheduling methodology, based on creation of bifurcations of codimension greater than one, is described in this paper. It is applied to a sample configuration that can usually be flown only to incidences a little above 20°, and used to show how a single additional pitch-axis control effector can be scheduled to modify the bifurcationary behaviour of the dynamics, and thus create a stable symmetric operating region at angles of attack in the vicinity of 40–50°.

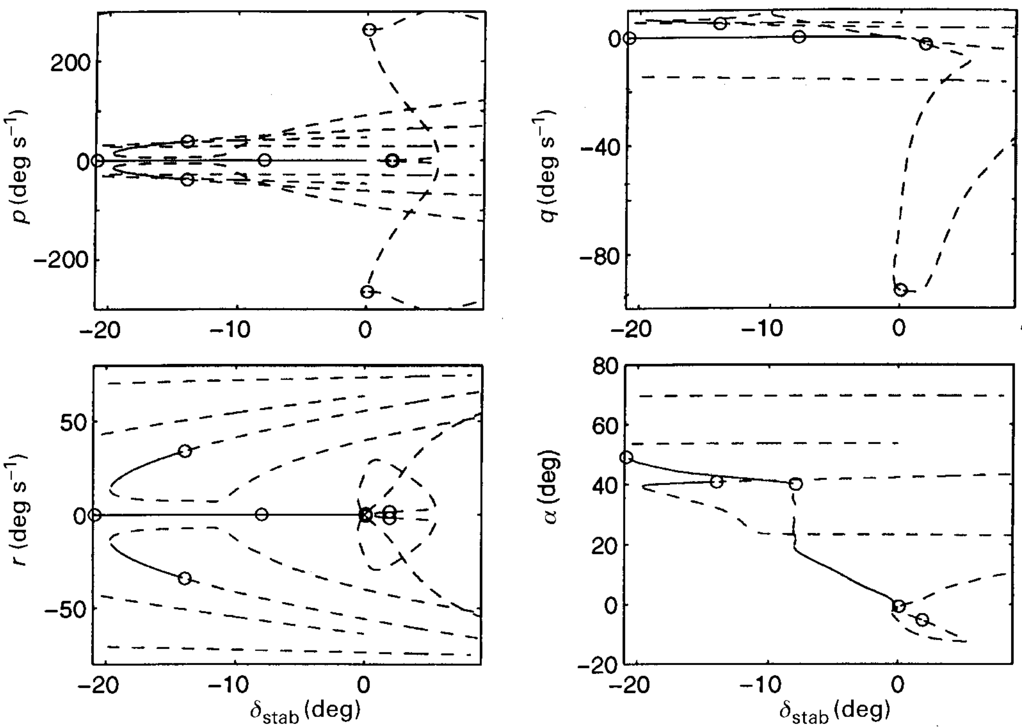

Figure 10. F-4J model equilibria: δ_ζ scheduled, all branches.

It is recommended that the methodology be tested on other aircraft configurations, preferably using models more appropriate to modern designs. The method offers scope for extension to more complex coupled situations, and to bifurcations from periodic orbits. The computational expense in applying it to the latter would be significantly greater than in the point attractor example demonstrated here. Investigation should also be conducted into the potential for the technique to be used not only to manipulate bifurcationary and stability properties but also to modify basins of attraction (which provide valuable information in the study of competing attractors).

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